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SOLUTION OF INVERSE PROBLEM OF SOLIDIFICATION OF A CYLINDRICAL INGOT

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An inverse problem of heat conduction that involves determination of the density of boundary heat flux providing a prescribed velocity of solidification front motion is solved by an integral method.

A Mathematical Model of Cylindrical Ingot Solidification. The solidification of a cylindrical ingot, a siagram of which is given in Fig. 1, is analyzed in a one-dimensional formulation. It is assumed that at the initial instant the entire cylindrical region is in a liquid state at the melting point T_{ph} and during solidification a liquid phase is maintained at this temperature $T_1(r, t) = T_{\mathrm{ph}}$. An unknown time-variable heat flux with a density $q(t)$ that provides the prescribed velocity $\xi'(t)$ of motion of the phase interface $r = \xi(t)$ inside the ingot is removed through the boundary surface $r = R$. The thermophysical properties of the solid phase of the ingot are assumed to be constant.

Under the conditions mentioned, the temperature field in a cylindrical layer of solidified material is described by the equation of nonstationary heat conduction

$$\frac{\partial T_2}{\partial t} = \frac{a_2}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_2}{\partial r} \right), \quad R - \xi(t) \le r \le R, \quad \xi(0) = 0 \tag{1}$$

with the boundary conditions

$$T_2(R - \xi, t) = T_{\mathrm{ph}}, \quad t \ge 0; \tag{2}$$

$$\lambda_2 \left. \frac{\partial T_2}{\partial r} \right|_{r = R - \xi} = - \rho_2 L \xi'(t), \quad t > 0; \tag{3}$$

$$\lambda_2 \left. \frac{\partial T_2}{\partial r} \right|_{r = R} = - q(t), \quad t > 0. \tag{4}$$

Analysis of regorous mathematical model (1)-(4) is difficult, due to its nonlinearity. Moreover, a strict solution in the cases when it can be obtained in the form of an infinite series is unsuitable for studying the considered physical process. By virtue of this we use an integral method [1] which allows us to find in closed analytical form a solution of (1)-(4) that is approximate but satisfactory from an engineering point of view. Averaging Eq. (1) within the limits from $r = R - \xi(t)$ to $r = R$ and allowing for conditions (2)-(4), we obtain the following integral of heat balance

$$\Theta' = \left(T_{\mathrm{ph}} + \frac{L}{c_2} \right) (R - \xi) \xi' - \frac{R}{c_2 \rho_2} q, \tag{5}$$

where

$$\Theta(t) = \int_{R-\xi}^{R} r T_2(r, t)\, dr. \tag{6}$$

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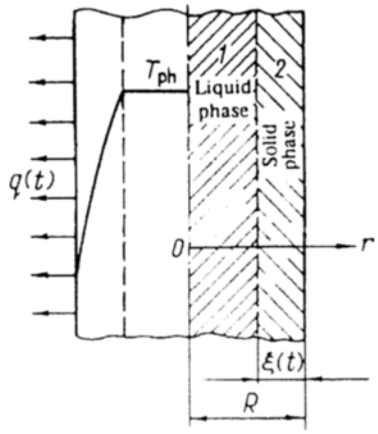

Fig. 1. Diagram of cylindrical ingot for internal problem of solidification.

In solving heat-conduction problems by the integral method one must assign a temperature profile. In [1] the question of an expedient choice of a temperature profile in solving problems of heat conduction in different regions. In particular, it is shown that in a region with cylindrical symmetry it has the form $T_2(r, t) = P_n(r) \ln (r)$, where $P_n(r)$ is a polynomial in r with coefficients dependent on t. In just this way the temperature profile is presented in [2], thus leading to a solution containing a singularity at $\xi(t) = R$. On the other hand, it is easily seen that the following function satisfies Eq. (1)

$$T_2(r, t) = A + B \left[-\mathrm{Ei} \left(-\frac{r^2}{4a_2 t} \right) \right], \tag{7}$$

where A and B are arbitrary constants;

$$-\mathrm{Ei}(-z) = \int_z^\infty \frac{e^{-t}}{r} dt = -C - \ln z + \sum_{n=1}^\infty (-1)^{n+1} \frac{z^n}{n!\,n}; \tag{8}$$

$C = 0.57721566...$ is the Euler constant.

It follows from (7) that if we restrict ourselves to the first n terms of expansion (8), we obtain an approximate solution of Eq. (1) in the form $T_2(r, t) = P_n(r^2) + C \ln r^2$, where $P_n(r^2)$ is a polynomial in r^2 with coefficients dependent on t. In presenting the temperature profile in such a form, we require that the unknown solution satisfy the averaged equation, i.e., the integral of heat balance (1), rather than heat-conduction equation (1). Then the initial heat-conduction equation will be satisfied on the average.

Thus, we present the temperature profile in the form

$$T_2(r, t) = A_1(t) + A_2(t) r^2 + A_3(t) \ln r^2. \tag{9}$$

To find functions $A_i(t)$ it is necessary to asign three boundary conditions. In the inverse problem considered here boundary conditions (2)-(4) serve as these conditions, and the integral of heat balance (5) is used to obtain a first-order differential equation with respect to $q(t)$. In the case of the direct problem of cylindrical ingot solidification under the effect of the prescribed heat flux $q(t)$ condition (3) should be replaced by another one not involving $\xi(t)$ since the coefficients $A_i(t)$ will contain ξ' and the integral of heat balance (5) ξ'', while there is only one initial condition for $\xi(t)$: $\xi(0) = 0$. In [2] this condition is obtained by integrating (2) with respect to t:

$$-\frac{\partial T_2}{\partial r} \frac{d\xi}{dt} + \frac{\partial T_2}{\partial t} = 0 \quad \text{at} \quad r = R - \xi \tag{10}$$

and eliminating ξ' from (3) by (10):

$$\left(\frac{\partial T_2}{\partial r}\right)^2 = -\frac{\rho_2 L}{\lambda_2}\frac{\partial T_2}{\partial t} \quad \text{at} \quad r = R - \xi . \tag{11}$$

Eliminating finally $\partial T_2/\partial t$ from (11) by Eq. (1), we have

$$\left(\frac{\partial T_2}{\partial r}\right)^2 = -\frac{L}{c_2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) \quad \text{at} \quad r = R - \xi . \tag{12}$$

Expression (12) is a third necessary condition, which is used instead by (3).

Solution and Analysis of Results. Using conditions (2) and (4) we find the coefficients $A_1(t)$ and $A_3(t)$:

$$A_1(t) = T_{\text{ph}} + \frac{Rq}{2\lambda_2}\ln(R-\xi)^2 + [R^2\ln(R-\xi)^2 - (R-\xi)^2]\,A_2(t)\,; \tag{13}$$

$$A_3(t) = \frac{Rq}{2\lambda_2} - R^2 A_2(t)\,, \tag{14}$$

and the temperature profile

$$T_2(r,t) = T_{\text{ph}} + \frac{Rq}{2\lambda_2}\ln\left(\frac{R-\xi}{r}\right)^2 + \left[R^2\ln\left(\frac{R-\xi}{r}\right)^2 + r^2 - (R-\xi)^2\right]A_2(t)\,. \tag{15}$$

The coefficient $A_2(t)$ can be found by two ways: from condition (3) when solving the inverse problem and from condition (12) when solving both the direct and inverse problems.

From condition (3) we find

$$A_2(t) = \frac{\rho_2 L(R-\xi)\,\xi' - Rq}{2\lambda_2(2R-\xi)\,\xi}\,, \tag{16}$$

which leads to a temperature profile of the form

$$T_2(r,t) = T_{\text{ph}} + \frac{[\rho_2 L(R-\xi)\,\xi' - Rq]\,[r^2 - (R-\xi)]}{2\lambda_2(2R-\xi)\,\xi} +$$

$$+ \frac{R(R-\xi)\,[R\rho_2 L\xi - (R-\xi)\,q]}{2\lambda_2(2R-\xi)\,\xi}\ln\left(\frac{R-\xi}{r}\right). \tag{17}$$

The employment of condition (12) gives

$$A_2(t) = -\frac{z_1 + z_2 + (R-\xi)\sqrt{\lambda_2 L(2z_1 + z_2)}}{z_3}\,, \tag{18}$$

where

$$z_1 = Rc_2 q(2R-\xi)\,\xi\,; \quad z_2 = \lambda_2 L(R-\xi)^2\,; \quad z_3 = 2\lambda_2 c_2(2R-\xi)^2\,\xi^2\,. \tag{19}$$

Substituting (18) into (15), we obtain an expression for the temperature profile that is suitable for solving direct and inverse problems of cylindrical ingot solidification.

It should be noted that the choice of a temperature profile in the form of (9) leads to expression (15), where $A_2(t)$ is found from (16) or (18), which in contrast to [3] has no singularity at $\xi(t) = R$. Actually, when $\xi(t) \to R$, it follows from (15) that temperature distribution in a solidified ingot will have the form

$$T_2(r, t) = T_{ph} - \frac{q(t)}{2\lambda_2 R} r^2, \quad 0 \le r \le R. \tag{20}$$

Two approximate solutions of both the direct and inverse problems of heat conduction are suggested in the paper.

The solution at $A_2(t) \equiv 0$, as follows from (9), corresponds to a simplified model when in expansion (8) the two first terms are used: $-\mathrm{Ei}(-z) \approx -C - \ln z$.

It follws from (16) that $A_2(t) \equiv 0$, if

$$\rho_2 L (R - \xi) \xi' - Rq = 0. \tag{21}$$

Integrating (21) at $\xi(0) = 0$ we obtain a relation for determination of the solidification process

$$\xi(t) = R - \sqrt{\left(R^2 - \frac{2R}{\rho_2 L} \int_0^t q(\tau)\, d\tau\right)}. \tag{22}$$

On the other hand, in the case of the inverse problem, Eq. (21) is used to determine the unknown flow

$$q(t) = \frac{\rho_2 L}{R} (R - \xi) \xi'. \tag{23}$$

As particular cases of the solution of the simplified model suggested in the paper approximate solutions of the direct problem of cylindrical ingot solidification that are known from literature, are obtained under boundary conditions of the first and third kinds.

From (15) at $A_2(t) \equiv 0$ we obtain

$$T_2(r, t) = T_{ph} + \frac{Rq}{2\lambda_2} \left(\frac{R - \xi}{r}\right)^2. \tag{24}$$

Hence it follows that to maintain a constant temperature $T_2(R, t) = T_s$ on the surface $r = R$ the flux $q(t)$ should be equal to

$$q(t) = \frac{\lambda_2 (T_s - T_{ph})}{\ln \dfrac{R - \xi}{R}}. \tag{25}$$

Substituting this value of $q(t)$ into (21), we have

$$\rho_2 L (R - \xi) \xi' - \frac{\lambda_2 (T_s - T_{ph})}{\ln \dfrac{R - \xi}{R}} = 0 \tag{26}$$

or

$$dt = \frac{\rho_2 L}{\lambda_2 (T_s - T_{ph})} (R - \xi) \ln \frac{R - \xi}{R}\, d\xi. \tag{27}$$

Integrating (27) at $\xi(0) = 0$, we find

$$t = \frac{L \rho_2 R^2}{\lambda_2 (T_{ph} - T_s)} \left[\frac{1}{4}\left(1 - \frac{r_f^2}{R^2}\right) + \frac{r_f^2}{2R^2} \ln \frac{r_f}{R}\right], \tag{28}$$

where $r_f = R - \xi$.

Formula (28) coincides with the known Leibenson solution obtained by another method under boundary conditions of the first kind [4].

The solution of Seban and London for boundary conditions of the third kind [5] is also obtained from the simplified model. Actually, so that the condition

$$\lambda_2 \frac{\partial T_2}{\partial r}\bigg|_{r=R} = \alpha\,[T_2(R,t) - T_m],$$

is satisfied at the boundary, one should assign the flux $q(t)$ in the form

$$q(t) = \frac{\alpha\,(T_{ph} - T_m)}{1 - \dfrac{\alpha R}{\lambda_2} \ln \dfrac{R-\xi}{R}}. \tag{29}$$

Substitution of (29) into (21) leads to the following differential equation for the dependence of the phase interface motion on time

$$\rho_2 L \left(1 - \frac{\alpha R}{\lambda_2} \ln \frac{R-\xi}{R}\right)(R-\xi)\,\xi' = \alpha R\,(T_{ph} - T_m), \tag{30}$$

or

$$dt = \frac{\rho_2 L}{\alpha R\,(T_{ph} - T_m)} \left(1 - \frac{\alpha R}{\lambda_2} \ln \frac{R-\xi}{R}\right)(R-\xi)\,d\xi. \tag{31}$$

Integrating (31) at $\xi(0) = 0$, we obtain

$$t = \frac{L\rho_2 R^2}{\lambda_2\,(T_{ph} - T_m)} \left[\left(\frac{1}{4} + \frac{\lambda_2}{2\alpha R}\right)\left(1 - \frac{r_f^2}{R^2}\right) + \frac{r_f^2}{2R^2} \ln \frac{r_f}{R}\right]. \tag{32}$$

Formulas (22), (28), and (32) allow one to determine time t_{fin} of the completion of the process of round ingot solidification at $\xi(t) = R$. From (22), in particular, we have boundary conditions of the second kind :

1. If

$$q(t) = \frac{K\rho_2 L}{2R}\left(\frac{R}{\sqrt{t}} - K\right), \tag{33}$$

then $\xi = K\sqrt{t}$. This parabolic relationship between the solidified layer thickness and time is shown as the law of the square root and is widely employed in studies of the process of ingot solification [6]. The process of solidification is completed at $t_f = R^2/K^2$.

2. If $q(t)$ = const, then

$$\xi(t) = R - \sqrt{\left(R^2 - \frac{2Rq}{\rho_2 L} - t\right)} \tag{34}$$

and the solidification process stops at

$$t_f = \frac{R\rho_2 L}{2q}. \tag{35}$$

The accuracy of the solution of the approximate model can be estimated from the condition of using the first two terms in expansion (8): $-\mathrm{Ei}(-z) \approx -C - \ln z$. Since (8) is an alternating series, the error of this substitution does not exceed $z = r^2/4a_2 t = \mathrm{Fo}^{-1}$. Consequently, the error of the simplified model will be of the order of Fo^{-1}.

We consider a more strict solution of the problem that is obtained using the first three terms in expansion (8): $-\mathrm{Ei}(-z) \approx -C - \ln z + z$, substituting (15) and (6) and allowing for the three coefficients $A_i(t)$ $(i = 1, 2, 3)$. Performing this substitution and integrating we find

$$\Theta(t) = \frac{T_{\mathrm{ph}}(2R-\xi)\xi}{2} + \frac{Rq}{4\lambda_2}\left[(2R-\xi)\xi + R^2 \ln\left(\frac{R-\xi}{R}\right)^2\right] +$$

$$+ \left\{(2R-\xi)^2 \xi^2 + 2R^2\left[(2R-\xi)\xi + R^2 \ln\left(\frac{R-\xi}{R}\right)\right]\right\} \frac{A_2(t)}{4}. \tag{36}$$

In expression (36) the coefficient $A_2(t)$ can be in the form of (16) or (18). When solving the inverse problemm it is expedient to use formula (16), which leads to a simpler equation for q. To solve the direct problem, one should use (18), since (16) leads to a second-order differential equation with respect to ξ.

Substituting (16) into (36) and satisfying the integral of heat balance (5), we obtain a first-order linear differential equation for the boundary flux

$$q' + P(t)\, q = Q(t), \tag{37}$$

where the following notation is introduced

$$P(t) = -\frac{2\left\{4a_2 b_1^2 + b_3\,[b_1^2 + 2R^2(b_1 + R^2 b_2)]\right\}}{b_1\,[(2R^2 - b_1)\, b_1 + 2R^2 (R^2 - b_1)\, b_2]}; \tag{38}$$

$$Q(t) = \frac{\rho_2 L\left\{[b_1 b_4 - \xi'^2 (2R^2 - b_1)]\,[b_1^2 + 2R^2(b_1 + R^2 b_2)] - 4b_1^3 \xi'^2 - 8a_2 b_1^2 b_3\right\}}{R b_1\,[(2R^2 - b_1)\, b_1 + 2R^2 (R^2 - b_1)\, b_2]}; \tag{39}$$

$$b_1 = \xi(2R - \xi); \quad b_2 = \ln\left(\frac{R-\xi}{R}\right)^2; \quad b_3 = \xi'(R-\xi); \quad b_4 = \xi''(R-\xi). \tag{40}$$

Integrating Eq. (37), we find

$$q(t) = \left[q_0 + \int_0^t Q(\tau)\exp\left(\int_0^\tau P(r)\, dr\right) d\tau\right] \exp\left(-\int_0^t P(\tau)\, d\tau\right). \tag{41}$$

Formula (41) give a complete solution of the above-formulated inverse problem of heat conduction.

It should be noted that an approximate solution of the inverse problem obtained from (9) with allowance for two coefficients $A_1(t)$ and $A_3(t)$ does not require the assignment of the initial value of the flux $q_0 = q(0)$. The unknown boundary flux in this approximation is fully determined by the law of solidification front motion. In particular, we find from (23)

$$q_0 = \rho_2 L \xi'(0). \tag{42}$$

With a more strict solution of this problem for an unknown flux $q(t)$ we obtain first-order differential Eq. (37), to solve which one should assign $q_0 = q(0)$. In accordance with the conditions of the problem the density of the boundary heat flux q_0 should be determined from the value of the velocity of solidification front motion at $t = 0$.

This initial condition can be taken in the form

$$q_0 = \begin{cases} \rho_2 L \xi'(0), & \text{if } \xi'(0) < \infty; \\ A = \mathrm{const}, & \text{if } \xi'(0) = \infty. \end{cases} \tag{43}$$

Now we consider the direct problem within the framework of a more-rigorous model and introduce an equation for determination of the solidification front motion in time. For this purpose, differentiating (6) and allowing for (2), we find

$$\Theta' = \frac{d}{dt}\int_{R-\xi}^{R} rT_2(r, t)\,dr = \left[T_{\mathrm{ph}}(R-\xi) + \int_{R-\xi}^{R} \frac{\partial rT_2(r, t)}{\partial \xi}\,dr\right]\xi'. \tag{44}$$

Substitution of (44) into the integral of heat balance (5) leads to the following differential equation

$$\frac{R}{c_2\rho_2}q = \left[\frac{L}{c_2}(R-\xi) - \int_{R-\xi}^{R}\frac{\partial rT_2}{\partial \xi}\,dr\right]\xi' \tag{45}$$

or

$$dt = \frac{c_2\rho_2}{R}\,\frac{\left[\frac{L}{c_2}(R-\xi) - \int_{R-\xi}^{R}\frac{\partial rT_2}{\partial \xi}\,dr\right]d\xi}{q}, \tag{46}$$

whose integration yields

$$t = \frac{c_2\rho_2}{R}\int_{R-\xi}^{R}\frac{\left[\frac{L}{c_2}(R-s) - \int_{R-s}^{R}\frac{\partial rT_2}{\partial s}\,dr\right]ds}{q}\,d\xi. \tag{47}$$

The integral expression (47) is a solution of the direct problem of cylindrical ingot solidification under boundary conditions of the secod kind.

In [3] an approximate solution of the indicated problem of cylindrical ingot solidification under boundary conditions of the first kind is obtained. It is easy to show that this solution follows from (47) as a particular case, if in (47) a flux $q(t)$ is assigned that ensures constancy of the temperature at the boundary $r = R$. This flux is found from (17), in which it should be assumed that $r = R$ and $T_2(R, t) = T_s$.

Completing the consideration of a more-rigorous solution in which the first three terms in expansion (8) are used, one should note the error of this model will be $O(\mathrm{Fo}^{-2})$.

NOTATION

t, time; r, spatial coordinate; $T_2(r, t)$, temperature of solid phase; $a_2 = \lambda_2/c_2\rho_2$, thermal diffusivity; ρ_2, c_2, λ_2, density, heat capacity, and thermal conductivity of solid phase; T_{ph}, temperature of solidification; L, latent heat of solidification; R, radius of cylindrical ingot; $\theta(t)$, average temperature of solid phase; $q(t)$, density of boundary heat flux; $r = \xi(t)$, equation of phase interface; $\xi'(t)$, velocity of solidification front motion; K, constant of solidification; $-\mathrm{Ei}(-z)$, integro-exponential fucntion; α, coefficient of heat transfer.

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